Journal of Sound and Vibration (2000) **236**(1), 33–48
doi:10.1006/jsvi.2000.2983, available online at http://www.idealibrary.com on IDEAL®

ON THE CHANGES CAUSED BY AN UNDERLYING COUPLED ROTATIONAL MOTION ON THE NATURAL FREQUENCIES OF A MASS–SPRING SYSTEM

N. S. CLARKE AND E. A. MORGAN

Department of Engineering Mathematics, University of Newcastle upon Tyne, NE1 7RU, England.
E-mail: *n.s.clarke@ncl.ac.uk*

(*Received* 26 *July* 1999, *and in final form* 17 *January* 2000)

When a system of N masses, linked together by springs, is disturbed from its static equilibrium position, then it will vibrate in a manner characterized by the N natural frequencies of the system. Should the whole system be in rotation with constant rotation speed then these natural frequencies are all decreased by an amount depending upon the rotation rate. However, if the rotation speed is increased beyond a certain level then the motion will become "unstable", i.e., no longer vibrational. Only rotational speeds below this level are considered here. In this work, the system is mounted upon a turntable in such a manner that the masses may move only radially and the turntable is set rotating and the masses released. As the total angular momentum is conserved then the motions of the masses are coupled with the rotation of the turntable; that is, the rotation speed is no longer constant but is intimately linked to the motions taking place upon it. The effect on the natural frequencies of this coupling, and also of the initial positions and velocities when coupling is present are investigated. Two cases are pursued, one in which the displacements from the equilibrium position are "small" and the other where the coupling is "weak". In both cases, all the natural frequencies increase from their values at constant rotation; that is, the coupling is a stabilizing influence. Initially, a single mass system is considered in order to gain insight before the more general N-mass system is tackled. Damping is ignored throughout.

1. INTRODUCTION

The motions of systems of masses connected by springs has long been well understood and is often invoked as a model for more complicated mechanical systems. If these motions are taking place in a rotating frame of reference, with a constant rotation rate, then the analysis is modified only in minor detail, provided that the rotation rate is not so large as to cause the system to become "unstable".

Here an extension of this problem is considered, in that the rotation rate is to be coupled with the motions of the masses. This is achieved by taking the system to be mounted upon a turntable whose moment of inertia is not so large that its rotation rate is unaffected by what occurs upon it.

It is imagined that the turntable is "wound-up" to its initial rotation speed and then free of external influences, and of any damping, the mass system is released from its initial configuration with initial velocities. In this paper, attention is confined to the relatively simple case for which the masses can move without friction along the same radial line fixed in the turntable, such a constraint being provided by, say, a guide rail. A future paper will deal with the more complicated case in which both radial and transverse vibrations are possible.

As the masses move radially, their moments of inertia (about the centre of the turntable) will vary in time, and so, as the total angular momentum must remain constant, the rotation rate of the turntable must also vary to reflect this influence.

Corresponding to a given initial configuration and initial rotation rate there will be an equilibrium configuration and its associated equilibrium rotation rate. This state of affairs will be referred to as *rotational* equilibrium to distinguish it from the *static* equilibrium which occurs in the absence of rotation. If the masses are displaced from the rotational equilibrium position then they will vibrate, assuming that the equilibrium position is a “stable one”. (The question of stability is addressed as an outcome of the analysis rather than being the main consideration.) The aim of this work is to find out how the natural frequencies of vibration are modified by the influence of the (coupled) motion of the turntable. Other properties of the motions may be found if required but here we shall concentrate upon the natural frequencies.

In section 2, a single-mass system is used to illustrate some of the techniques to be used in the more general N-mass system. The equation which governs the displacement of the mass is a second order ordinary differential equation which is non-linear due to the coupling with the turntable. For an overview of such equations the reader may consult many suitable texts, such as Stoker [1] or Jordan and Smith [2]. It is possible to find a first integral of this equation (representing conservation of energy) but the authors have so far been unable to use this to advantage, and so have used the original equation to make progress towards their goal by means of a perturbation analysis.

There are four independent dimensionless parameters appearing in this problem whose individual smallnesses could in principle be used as a basis for a perturbation scheme. These are

(a) the dimensionless amplitude of vibration, i.e., the initial position is “close” to the equilibrium position and the dimensionless initial velocity is “small”.
(b) the ratio of moment of inertia of the mass (in say, its equilibrium position) to that of the turntable, i.e., weak coupling,
(c) the ratio of the rotation rate to the frequency of vibration in the static case, and
(d) dimensionless initial velocity (with the other parameters not being small).

Case (d) does not lead to any simplification, and nor does (c), although at first sight it appears promising but it leads to problems which can only be solved by numerical quadrature. This means that there is very little advantage in using this perturbation approach over a numerical method for the complete problem.

However, cases (a) and (b) are eminently suitable to be treated by perturbation methods and each is considered in section 2. In both cases, it is found that an *increase* in the small parameter leads to an *increase* in the natural frequency of vibration of the system.

In section 3, the N-mass problem is considered and the ideas developed in section 2 are used. Not only are the masses coupled to the turntable but also to each other and the governing equations are now N coupled second-order non-linear ordinary differential equations. It is not uncommon in linear problems to uncouple the governing equations by modal matrix techniques before solving. This technique is adapted here to uncouple the “linear part” of the equations before using a perturbation technique to overcome the difficulties presented by the non-linear part. The method of multiple scales is used to overcome the fact that the perturbation scheme is singular in the case of weak coupling.

As in the single-mass case, it is found that the natural frequencies all increase with increasing values of either of the small parameters (i.e., small amplitude or weak coupling).

Reassuringly, when both parameters are small the expressions derived for the natural frequencies coincide.

This work is concluded in section 4 by an illustration of a specific simple case (a two-mass system) to show how the detailed calculations may be carried out.

2. A SINGLE MASS SYSTEM

A mass m is mounted on a frictionless guide rail on a turntable which is free to rotate without friction about its centre. The position, relative to the centre, of the mass at time t will be denoted by $r(t)$, when the turntable is rotating with angular speed $\omega(t)$.

The radial acceleration, in the inertial frame, of the mass is

$$\ddot{r} - \omega^2 r$$

(dots denote differentiation with respect to time) and so by Newton's law the equation governing the motion of the mass is

$$\ddot{r} - \omega^2 r = -A(r - \hat{r}). \quad (2.1)$$

Here $\hat{r}$ is the static equilibrium position (i.e., for the non-rotating, non-oscillating case) and A is the system constant. The following examples may shed some light:

(a) If the mass is attached to the centre by a spring of spring constant k and natural length l then $A = k/m$ and $\hat{r} = l$.

(b) If the mass is attached to the centre by a spring of spring constant k_1 and natural length l_1 and also to the edge of the turntable (distance L from the centre) by a spring of spring constant k_2 and natural length l_2 then

$$A = (k_1 + k_2)/m \quad \text{and} \quad \hat{r} = \{k_1 l_1 + k_2(L - l_2)\}/(k_1 + k_2).$$

For this case see Figure 1.

It should be noted that the less specific form of equation (2.1) is adopted rather than one of the above examples so that the transition to an N-mass system is more seamless in section 3.

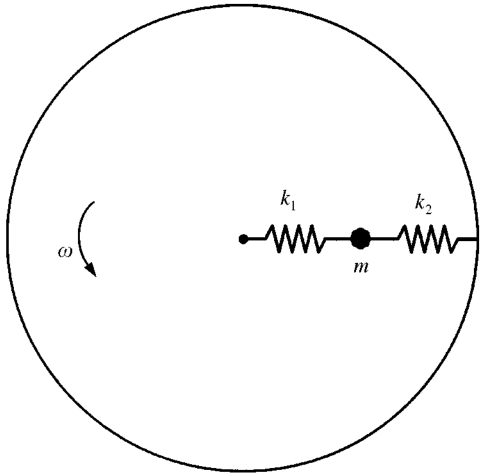

Figure 1. Layout of single-mass system as given in (b) above.

It seems likely that there is a rotating equilibrium position in which the spring forces just balance the centripetal forces and so r and ω are constant. If the equilibrium position and rotation rate are denoted by R and Ω then from equation (2.1)

$$R = \hat{r}A/(A - \Omega^2). \tag{2.2}$$

From this it is evident that it is necessary that

$$\Omega^2 < A,$$

and there will be other possible restrictions so that the mass is confined to the turntable.

In addition to equation (2.1) the condition is also imposed that the angular momentum H of the whole system must remain constant during the motion, that is

$$H = (I + mr^2)\,\omega = (I + mR^2)\,\Omega = (I + mr^{(0)2})\,\omega^{(0)}, \tag{2.3}$$

where I is the moment of inertia of the turntable and $r^{(0)}$ and $\omega^{(0)}$ are the initial position and rotation speed respectively. In order to solve the full problem the initial radial velocity $u^{(0)}$ of the mass will also have to be prescribed.

Note that from equation (2.2) and the second of equation (2.3) R and Ω may be found, given the other system parameters, although this could be achieved only numerically or approximately as exact expressions cannot be found. It seems intuitively obvious that there can only be one rotational equilibrium position and rotation speed, and this is indeed so, as is shown in Appendix A.

With ω given by equation (2.3) put into equation (2.1) a non-linear equation for $r(t)$ is obtained. A first integral of this equation (i.e., the energy equation) may be readily found, and this in turn may be integrated numerically for any specific values of the parameters. However, a more limited range of values of these parameters is preferred, and the values kept within that range. In this way it can be seen how the parameters influence the vibrations. For this purpose, by way of illustration, the natural frequency of vibration will be concentrated upon and two restricted parameter domains considered as follows.

2.1. SMALL VIBRATIONS ABOUT EQUILIBRIUM

For this case to occur obviously the initial position $r^{(0)}$ and initial rotation rate $\omega^{(0)}$ are required to be "close" to their equilibrium values R and Ω, (and also that the initial velocity $u^{(0)}$ is "small" is implied). A small parameter δ is therefore defined by

$$\delta = (r^{(0)} - R)/R \quad (\ll 1), \tag{2.4}$$

and

$$r = R\{1 + \delta X(t) + 0(\delta^2)\}$$

and

$$\omega = \Omega\{1 + \delta W(t) + 0(\delta^2)\}. \tag{2.5}$$

Forms (2.5) are substituted into equations (2.1) and (2.3) and only the terms of $0(\delta)$ are retained (i.e., linearizing about the equilibrium position). From equation (2.1), and by use of equation (2.2)

$$\ddot{X} + (A - \Omega^2)X - 2\Omega^2 W = 0,$$

and from equation (2.3)

$$W = -2\frac{\varepsilon}{1+\varepsilon}X,$$

where

$$\varepsilon = mR^2/I. \tag{2.6}$$

Here ε is not necessarily small (though see the next section). Hence, the equation governing the vibration is

$$\ddot{X} + \left\{A - \Omega^2\frac{(1-3\varepsilon)}{1+\varepsilon}\right\}X = 0, \tag{2.7}$$

and this means that the natural frequency of vibration is

$$\frac{1}{2\pi}\left\{A - \Omega^2\frac{(1-3\varepsilon)}{1+\varepsilon}\right\}^{1/2}. \tag{2.8}$$

From this for $\varepsilon \to 0$ (i.e., a massive turntable) the rotation rate is constant and the natural frequency is lowered from its static value. For $\varepsilon > 0$ the natural frequency rises again and indeed for a "lightweight" turntable (one for which $I < 3mR^2$) the natural frequency will be greater than its static value. This is of course due to the coupling of the motions. Curiously, if $I = 3mR^2$ then the natural frequency is independent of the rotation rate (to this order).

2.2. VIBRATIONS WITH WEAK COUPLING

For this case, the small parameter introduced is

$$\varepsilon = mR^2/I,$$

as in equation (2.6), but now considered small, and so perturbations about a constant rotation speed are considered. Here, however, the initial position need not be "close" to the equilibrium position and the vibrations need not be small (except in the sense of keeping the spring(s) within the linear response, and also keeping the mass on the turntable).

From equation (2.3) it may be written that

$$\omega = \Omega(I + mR^2)/(I + mr^2)$$
$$= \Omega(1+\varepsilon)/(1+\varepsilon r^2/R^2), \tag{2.9}$$

and so from equation (2.1) the following governing equation is obtained

$$\ddot{r} + r\{A - \Omega^2(1+\varepsilon)^2/(1+\varepsilon r^2/R^2)^2\} = A\hat{r}. \tag{2.10}$$

In addition, the initial position and velocity is taken to be

$$r(0) = r^{(0)}, \qquad \dot{r}(0) = u^{(0)}. \tag{2.11}$$

If a perturbation scheme is attempted, based upon small ε, for equation (2.10) then it is found that the scheme is singular and so the method of multiple scales is used to overcome this difficulty (see, for example, reference [3]). Retaining t as a fast variable, introduce τ as a slow variable

$$\tau = \varepsilon t.$$

As the expansion is expanding only to $0(\varepsilon)$ further slow variables are not required for this problem. The dependent variable $r(t, \varepsilon)$ is then regarded as $r(t, \tau, \varepsilon)$ and so satisfies the equation.

$$\frac{\partial^2 r}{\partial t^2} + r\{A - \Omega^2(1+\varepsilon)^2(1+\varepsilon r^2/R^2)^{-2}\} = A\hat{r} - \varepsilon 2\,\frac{\partial^2 r}{\partial t\,\partial \tau} + 0(\varepsilon^2). \tag{2.12}$$

A solution is sought in the form

$$r(t, \tau, \varepsilon) = r^{(1)}(t, \tau) + \varepsilon r^{(2)}(t, \tau) + 0(\varepsilon^2), \tag{2.13}$$

and this leads to a sequence of problems, namely

$$\frac{\partial^2 r^{(1)}}{\partial t^2} + (A - \Omega^2)\,r^{(1)} = A\hat{r} \tag{2.14}$$

with

$$r^{(1)}(0,0) = r^{(0)}, \qquad \frac{\partial r^{(1)}}{\partial t}(0,0) = u^{(0)}$$

and

$$\frac{\partial^2 r^{(2)}}{\partial t^2} + (A - \Omega^2)\,r^{(2)} = 2\Omega^2 r^{(1)}(1 - r^{(1)2}/R^2) - 2\,\frac{\partial^2 r^{(1)}}{\partial t\,\partial \tau}, \tag{2.15}$$

together with initial conditions which will not be used.

The solution of equation (2.14) is

$$r^{(1)} = R + B(\tau)\cos\mu t + C(\tau)\sin\mu t$$

with

$$B(0) = r^{(0)} - R \quad \text{and} \quad C(0) = u^{(0)}/\mu, \tag{2.16}$$

where

$$\mu^2 = A - \Omega^2.$$

This form for $r^{(1)}$ is now put into the right-hand side of equation (2.15) and to avoid secular terms appearing in the solution for $r^{(2)}$ there must be no terms proportional to either $\sin\mu t$ or $\cos\mu t$. For this to be so B and C must be chosen so that

$$B' - \mu\sigma C = 0$$

and

$$C' + \mu\sigma B = 0, \tag{2.17}$$

where

$$\sigma = \Omega^2\left\{2R^2 + \frac{3}{4}(B^2 + C^2)\right\}\Big/\mu^2 R^2, \tag{2.18}$$

and is plainly positive. It immediately follows from equation (2.17) that $B^2 + C^2$ is constant and, from the initial conditions that

$$B^2 + C^2 = (r^{(0)} - R)^2 + u^{(0)2}/\mu^2,$$

and so

$$\sigma = \frac{\Omega^2}{A - \Omega^2}\left\{2 + \frac{3}{4}\left(1 - \frac{r^{(0)}}{R}\right)^2 + \frac{3}{4}\,\frac{u^{(0)2}}{R^2(A - \Omega^2)}\right\}. \tag{2.19}$$

The solution (2.16) may now be written as

$$r^{(1)} = R + (r^{(0)} - R)\cos\mu(1+\varepsilon\sigma)t + \frac{u^{(0)}}{\mu}\sin\mu(1+\varepsilon\sigma)t, \tag{2.20}$$

and so the natural frequency is

$$\frac{1}{2\pi}\sqrt{A-\Omega^2}\{1+\varepsilon\sigma+0(\varepsilon^2)\}, \tag{2.21}$$

which is dependent upon the initial conditions, this being a manifestation of the non-linearity of the system.

It immediately follows that, as $\sigma > 0$, the natural frequency is always greater than its "equilibrium" value whenever coupling is present.

In the region of overlap of the two cases considered in this section, $r^{(0)} = R(1+0(\delta))$ and $u^{(0)} = 0(\delta)$. In both cases, then the expression for the natural frequency is given by

$$\frac{1}{2\pi}(A-\Omega^2)^{1/2}\left\{1+\varepsilon\frac{2\Omega^2}{A-\Omega^2}+\cdots\right\}.$$

However, if the vibrations are not small then as can be seen from equation (2.19) the natural frequency will depend upon the initial position and velocity.

Earlier in this section, it was mentioned that Ω^2 must certainly be less than A in order for the basic motion to be stable, and also that there would be other practical limitations on the rotation speed, such as keeping the mass on the turntable. It is now possible to consider this further. From equation (2.20) the maximum displacement is given by $R + \{(r^{(0)}-R)^2 + u^{(0)2}/\mu^2\}^{1/2}$, and this must be less than L, the radius of the turntable. This leads to the condition (by use of equation (2.2)) that

$$\Omega^2 < A\left(1-\frac{2\hat{r}}{L+r^{(0)}}\right) - \frac{u^{(0)2}}{L^2 - r^{(0)2}}$$

for the motion to be meaningful.

3. AN N-MASS SYSTEM

In this section, the problem of the previous section is generalized to the extent of allowing for N-masses, interconnected by springs, to vibrate along the same radial line, with the motion of each mass being coupled with that of the other masses and with the turntable.

Let $r_i(t)$ be the position of mass m_i at time t and $\hat{r}_i$ be its position in the static situation, for $i = 1, 2, \ldots, N$. Similarly, the initial position and radial velocity are denoted by $r_i^{(0)}$ and $u_i^{(0)}$. It is evident that

$$r_i < r_j \quad \text{for } i < j,$$

as the masses are ordered outwards from the centre. The initial rotation rate of the turntable will also be denoted by $\omega^{(0)}$, whilst at a general time by $\omega(t)$.

Let $\mathbf{A}$ be the system matrix (with components A_{ij}) generated by the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{S}$ via $\mathbf{A} = \mathbf{M}^{-1}\cdot\mathbf{S}$. It is taken that $\mathbf{A}$ is known, along with its eigenvalues λ_i^2 and corresponding eigenvectors $\mathbf{p}_i$. (For an account of matrix methods in vibration analysis see, for example, reference [4] or reference [5].)

Newton's law is applied to each mass and the governing equations, in matrix form, are

$$\ddot{\mathbf{r}} - \omega^2 \mathbf{r} = -\mathbf{A} \cdot (\mathbf{r} - \hat{\mathbf{r}}), \tag{3.1}$$

$\mathbf{r}$ being the column matrix $\{r_i\}$, etc. It is worth noting that the formulation adopted here will also allow configurations more general than already indicated to be dealt with in that several mass–spring systems on different radial lines could be obtained provided that there was no direct connection between masses on different lines (i.e., the springs only act radially). The motions of masses on different radial lines are coupled through the motion of the turntable only. These different layouts would manifest themselves through the system matrix $\mathbf{A}$.

It can be seen from equation (3.1) that there will be an equilibrium rotation rate Ω corresponding to a rotational equilibrium position configuration $\mathbf{R}$, given by

$$\mathbf{R} = (\mathbf{A} - \Omega^2 \mathbf{I})^{-1} \cdot \mathbf{A} \cdot \hat{\mathbf{r}}. \tag{3.2}$$

For this to be realistic Ω^2 must be smaller than all the eigenvalues of $\mathbf{A}$.

Also conservation of angular momentum yields

$$(I + \sum m_j r_j^2)\, \omega = (I + \sum m_j r_j^{(0)2})\, \omega^{(0)} = (I + \sum m_j R_j^2)\, \Omega = H, \tag{3.3}$$

which is constant. Here, and in what follows, the summation symbol implies summation over all possible values of the suffix, unless otherwise stated. It should be noted that Ω and $\mathbf{R}$ may be obtained from equations (3.2) and (3.3), at least numerically, for given values of the other parameters.

Equation (3.1) is a system of coupled differential equations which are non-linear due to the rotational effects. Before proceeding to use perturbation methods to obtain asymptotic solutions it is profitable to uncouple the linear part of equation (3.1). The modal matrix $\mathbf{P}$ has as its columns $\mathbf{p}_i$ the eigenvectors of $\mathbf{A}$ (i.e. $(\mathbf{p}_i)_j = P_{ji}$) and is used to transform $\mathbf{r}$ into $\boldsymbol{\zeta}$ via

$$\mathbf{r} = \mathbf{P} \cdot \boldsymbol{\zeta}. \tag{3.4}$$

At the same time let $\mathbf{r}^{(0)} = \mathbf{P} \cdot \boldsymbol{\zeta}^{(0)}$, $\hat{\mathbf{r}} = \mathbf{P} \cdot \hat{\boldsymbol{\zeta}}$, $\mathbf{R} = \mathbf{P} \cdot \boldsymbol{\zeta}$ and $\mathbf{u}^{(0)} = \mathbf{P} \cdot \mathbf{v}^{(0)}$.

By construction $\mathbf{P}^{-1} \cdot \mathbf{A} \cdot \mathbf{P}$ is diagonal with elements λ_i^2.

It is also convenient to introduce the generalized masses M_i defined by

$$M_i = \mathbf{p}_i^{\mathrm{T}} \cdot \mathbf{M} \cdot \mathbf{p}_i, \tag{3.5}$$

from which it follows that the $M_i > 0$ as $\mathbf{M}$ is positive definite. It may then be shown that

$$\sum m_j P_{ji} P_{jk} = M_i \delta_{ik}, \tag{3.6}$$

and so

$$\sum m_j r_j^2 = \sum M_j \zeta_j^2. \tag{3.7}$$

In terms of the new variables ζ_i equation (3.1) may be written in partially uncoupled form (note that ω depends upon all the ζ_j)

$$\ddot{\zeta}_i + (\lambda_i^2 - \omega^2)\, \zeta_i = \lambda_i^2 \hat{\zeta}_i. \tag{3.8}$$

Similarly, equations (3.2) and (3.3) become

$$\xi_i = \frac{\lambda_i^2}{(\lambda_i^2 - \Omega^2)}\hat{\zeta}_i, \tag{3.9}$$

and

$$(I + \sum M_j \zeta_j^2)\,\omega = (I + \sum M_j \zeta_j^{(0)2})\omega^{(0)} = (I + \sum M_j \xi_j^2)\Omega = H, \tag{3.10}$$

with initial conditions

$$\zeta_i(0) = \zeta_i^{(0)} \quad \text{and} \quad \dot{\zeta}_i(0) = v_i^{(0)}. \tag{3.11}$$

As in the single mass case two perturbation schemes will be considered, one based upon small amplitude vibrations about the equilibrium position, and the other based upon weak coupling.

3.1. SMALL VIBRATIONS ABOUT THE EQUILIBRIUM POSITION

A small parameter δ is introduced, which is a measure of the relative size of the amplitudes, for instance

$$\delta = \sum (r_i^{(0)} - R_i)/\sum R_i,$$

though the precise definition of δ is not relevant. Letting

$$\zeta_i = \xi_i(1 + \delta X_i + 0(\delta^2)), \quad \omega = \Omega(1 + \delta W + 0(\delta^2)),$$

and using equation (3.9),

$$\ddot{X}_i + (\lambda_i^2 - \Omega^2)X_i - 2\Omega^2 W = 0,$$

can be obtained from equation (3.8) (retaining only terms of order δ) and

$$W = -\frac{2\Omega}{H}\sum M_j \xi_j^2 X_j$$

from equation (3.10).

These two may be combined to yield

$$\ddot{\mathbf{X}} + \mathbf{\Phi}\cdot\mathbf{X} = \mathbf{0},$$

where the matrix $\mathbf{\Phi}$ is given by

$$\mathbf{\Phi}_{ij} = (\lambda_i^2 - \Omega^2)\,\delta_{ij} + 4\,\frac{\Omega^3}{H}\,M_j\xi_j^2. \tag{3.12}$$

Hence, the natural frequencies are given by $\phi_j/2\pi$ where ϕ_j^2 are the eigenvalues of $\mathbf{\Phi}$.

It is not difficult to show that each of these eigenvalues ϕ_j^2 is greater than its uncoupled counterpart $(\lambda_j^2 - \Omega^2)$, but less than $(\lambda_{j+1}^2 - \Omega^2)$ (assuming that the static eigenvalues λ_j^2 have been ordered) (see Appendix B).

In order to compare the results given here by equation (3.12) with those of the next case in the overlap region where both should be valid, not only are the displacements taken to be close to the equilibrium positions but also $I \gg \sum M_j \xi_j^2$. This means that H is approximated by $I\Omega$ and the natural frequencies will then be given by

$$\frac{1}{2\pi}\phi_i \approx \frac{1}{2\pi}\sqrt{\lambda_i^2 - \Omega^2}\left\{1 + 2\frac{\Omega^2}{I}\frac{M_i \xi_i^2}{(\lambda_i^2 - \Omega^2)}\right\}.$$

This shows that each natural frequency increases from its "equilibrium" value due to the coupling.

3.2. VIBRATIONS WITH WEAK COUPLING

For this case, the moment of inertia of the turntable is much greater than that of all the masses combined. Therefore, a small dimensionless parameter is introduced, defined by

$$\begin{aligned} \varepsilon &= \sum m_j R_j^2 / I \\ &= \sum M_j \xi_j^2 / I. \end{aligned} \tag{3.13}$$

From equation (3.10) this means that

$$\omega = \Omega(1 + \varepsilon)(1 + \varepsilon \sum M_j \zeta_j^2 / \sum M_j \xi_j^2)^{-1},$$

i.e.,

$$\omega \approx \Omega\{1 + \varepsilon[1 - \sum M_j \zeta_j^2 / \sum M_j \xi_j^2] + 0(\varepsilon^2)\}. \tag{3.14}$$

It is concluded that the turntable rotates with almost constant speed, and so the initial positions rather than the equilibrium positions could equally well have been used in the definition of ε, which would have changed equation (3.13) in detail but not in spirit. Note that even though the rotation speed remains close to its equilibrium value this is not necessarily true for the displacements.

When equation (3.13) is used in equation (3.8)

$$\ddot{\zeta}_i + (\lambda_i^2 - \Omega^2)(\zeta_i - \xi_i) = \varepsilon 2\Omega^2 \zeta_i\{1 - \sum M_j \zeta_j^2 / \sum M_j \xi_j^2\} + 0(\varepsilon^2). \tag{3.15}$$

Solutions to this equation and the initial conditions (3.11) are now sought in the form of a perturbation expansion in the parameter ε. Unless evasive action is taken, secular terms will arise at the order ε stage and so the method of multiple scales will be used. To this end t is regarded as the "fast" time variable and

$$\tau = \varepsilon t,$$

to be the "slow" time variable. The variables ζ_i are regarded as functions of t, τ and ε, and solutions in the form

$$\zeta_i(t, \tau; \varepsilon) = \zeta_t^{(1)}(t, \tau) + \varepsilon \zeta_i^{(2)}(t, \tau) + 0(\varepsilon^2), \tag{3.16}$$

are sought for equation (3.15) which now becomes

$$\frac{\partial^2 \zeta_i}{\partial t^2} + \mu_i^2(\zeta_i - \xi_i) = \varepsilon\left(-2\frac{\partial^2 \zeta_i}{\partial t\, \partial \tau} - 2\Omega^2 \zeta_i[1 - \alpha \sum M_j \zeta_j^2]\right) + 0(\varepsilon^2), \tag{3.17}$$

where

$$\mu_i^2 = \lambda_i^2 - \Omega^2,$$

and

$$\alpha = 1/\sum M_j \xi_j^2 = 1/\sum m_j R_j^2,$$

are introduced for temporary convenience.

With equation (3.16) put into equations (3.17) and (3.11) it is found that the problem for the first order variables $\zeta_i^{(1)}$ is

$$\frac{\partial^2 \zeta_i^{(1)}}{\partial t^2} + \mu_i^2(\zeta_i^{(1)} - \xi_i) = 0$$

with

$$\zeta_i^{(1)}(0,0) = \zeta_i^{(0)} \quad \text{and} \quad \frac{\partial \zeta_i^{(1)}}{\partial t}(0,0) = v_i^{(0)},$$

the solution to which is

$$\zeta_i^{(1)} = \xi_i + B_i(\tau)\cos\mu_i t + C_i(\tau)\sin\mu_i t,$$

where

$$B_i(0) = \zeta_i^{(0)} - \xi_i \quad \text{and} \quad C_i(0) = v_i^{(0)}/\mu_i. \tag{3.18}$$

The problem for the second order variables $\zeta_i^{(2)}$ is

$$\frac{\partial^2 \zeta_i^2}{\partial t^2} + \mu_i^{(2)}\zeta_i^2 = -2\frac{\partial^2 \zeta_i^{(1)}}{\partial t\,\partial\tau} + 2\Omega^2\zeta_i^{(1)}(1 - \alpha\sum M_j \zeta_j^{(1)2}), \tag{3.19}$$

together with initial conditions which will not be used. To avoid secular terms arising in the solution for $\zeta_i^{(2)}$ it must be ensured that on the right-hand side of equation (3.19) there are no terms proportional to $\sin\mu_i t$ or $\cos\mu_i t$. This condition leads to

$$\frac{\mathrm{d}B_i}{\mathrm{d}\tau} - \mu_i\sigma_i C_i = 0$$

and

$$\frac{\mathrm{d}C_i}{\mathrm{d}\tau} + \mu_i\sigma_i B_i = 0, \tag{3.20}$$

where

$$\mu_i^2\sigma_i = \Omega^2\alpha\left\{2M_i\xi_i^2 + \frac{1}{4}M_i(B_i^2 + C_i^2) + \frac{1}{2}\sum M_j(B_j^2 + C_j^2)\right\}.$$

Note that $\sigma_i > 0$ for all i. Equations (3.20) indicate that $B_i^2 + C_i^2$ is constant and so from the initial values in equation (3.18)

$$B_i^2 + C_i^2 = (\zeta_i^{(0)} - \xi_i)^2 + v^{(0)2}/\mu_i^2,$$

and

$$\sigma_i = \Omega^2\left\{2M_i\xi_i^2 + \frac{1}{4}M_i[(\zeta_i^{(0)} - \xi_i)^2 + v_i^{(0)2}/\mu_i^2] + \frac{1}{2}\sum M_j[(\zeta_j^{(0)} - \xi_j)^2 + v_j^{(0)}/\mu_j^2]\right\}\Big/\mu_i^2\sum M_j\xi_j^2. \tag{3.21}$$

It also follows from equation (3.20) that

$$B_i(\tau) = (\zeta_i^{(0)} - \xi_i)\cos\mu_i\sigma_i\tau + (v_i^{(0)}/\mu_i)\sin\mu_i\sigma_i\tau$$

and

$$C_i(\tau) = -(\zeta_i^{(0)} - \xi_i)\sin\mu_i\sigma_i\tau + (v_i^{(0)}/\mu_i)\cos\mu_i\sigma_i\tau,$$

and so

$$\zeta_i^{(1)} = \xi_i + (\zeta_i^{(0)} - \xi_i)\cos\mu_i(1+\varepsilon\sigma_i)t + (v_i^{(0)}/\mu_i)\sin\mu_i(1+\varepsilon\sigma_i)\,t. \tag{3.22}$$

As can be seen from equation (3.22) *all* the natural frequencies increase as a result of the coupling, as the $\sigma_i > 0$.

The natural frequencies are given by

$$\frac{1}{2\pi}\mu_i[1 + \varepsilon\sigma_i + 0(\varepsilon^2)],$$

and to compare this with the value given in the case of small vibrations it is essential that $\zeta_i^{(0)} \approx \xi_i$ and $v_i^{(0)} \approx 0$ whence

$$\sigma_i = 2\Omega^2 M_i\xi_i^2/\mu_i^2 \sum M_j\xi_j^2.$$

This means that in this dual limiting sense the natural frequencies are given approximately by

$$\frac{1}{2\pi}\sqrt{\lambda_i^2 - \Omega^2}\left\{1 + 2\,\frac{\Omega^2}{I}\,\frac{M_i\xi_i^2}{(\lambda_i^2 - \Omega^2)}\right\},$$

which agrees with equation (3.13).

If the starting positions are not close to the equilibrium values then, as equation (3.21) indicates, the natural frequencies will depend upon the initial values of the positions and velocities, both of which will tend to increase the natural frequencies.

4. ILLUSTRATION BY WAY OF A PARTICULAR TWO-MASS SYSTEM FOR WEAK COUPLING

Consider the system of two masses, both of mass m, connected to each other by a spring of spring constant k. The inner mass is connected to the axis and the outer mass to the perimeter of the turntable by springs, both of spring constant k. The simplicity of this system has been chosen so that the detail does not unduly obscure the essence of the procedure, and it is illustrated in Figure 2.

It is simple to show that the system matrix is

$$\mathbf{A} = \frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.1}$$

and the static equilibrium positions are obviously

$$\hat{r}_1 = \tfrac{1}{3}L, \qquad \hat{r}_2 = \tfrac{2}{3}L$$

with L being the radius of the turntable.

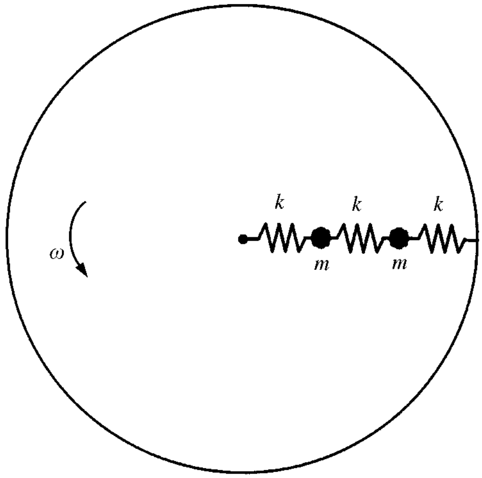

Figure 2. A two-mass system.

The masses are held in this position, the turntable wound up to angular speed $\omega^{(0)}$ and the masses released from rest. This gives

$$r_1^{(0)} = \tfrac{1}{3}L, \quad r_2^{(0)} = \tfrac{2}{3}L, \quad u_1^{(0)} = u_2^{(0)} = 0.$$

The rotational equilibrium positions are given by

$$R_1 = \frac{1}{(1-\gamma)(3-\gamma)}L, \quad R_2 = \frac{2-\gamma}{(1-\gamma)(3-\gamma)}L, \tag{4.2}$$

where $\gamma = m\Omega^2/k$ and in terms of the initial rotation rate the conservation of angular momentum may be used to give the equilibrium rotation rate Ω as

$$\Omega \equiv \omega^{(0)}\left\{1 - \varepsilon\left[1 - \frac{5}{9}\frac{(1-\gamma^{(0)})^2(3-\gamma^{(0)})^2}{(1+(2-\gamma^{(0)})^2)}\right] + 0(\varepsilon^2)\right\},$$

where $\gamma^{(0)} = m\omega^{(0)2}/k$.

For stability it is necessary for $\gamma < 1$ and to keep the masses on the turntable it is necessary (but not sufficient) for $R_2 < L$, which leads to the condition that

$$\gamma < \frac{(3-\sqrt{5})}{2} \quad (\approx 0{\cdot}382).$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1^2 = 3k/m$ and $\lambda_2^2 = k/m$, with corresponding eigenvectors $\mathbf{p}_1 = (-1, 1)^{\mathrm{T}}$ and $\mathbf{p}_2 = (1, 1)^{\mathrm{T}}$, and so the modal matrix is given by

$$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.3}$$

Using this matrix the generalized masses $M_1 = M_2 = 2m$ and the transformed positions are

$$\hat{\zeta}_1 = \zeta_1^{(0)} = \frac{1}{6}L, \quad \hat{\zeta}_2 = \zeta_2^{(0)} = \frac{1}{2}L,$$

and

$$\xi_1 = \frac{1}{2}\frac{L}{3-\gamma}, \quad \xi_2 = \frac{1}{2}\frac{L}{1-\gamma}, \tag{4.4}$$

and

$$\zeta_1 \approx \frac{1}{6}\frac{L}{3-\gamma}\{3 - \gamma\cos\mu_1(1+\varepsilon\sigma_1)t\},$$

$$\zeta_1 \approx \frac{1}{2}\frac{L}{1-\gamma}\{1 - \gamma\cos\mu_2(1+\varepsilon\sigma_2)t\}, \tag{4.5}$$

where

$$\mu_1^2 = \frac{k}{m}(3-\gamma), \quad \mu_2^2 = \frac{k}{m}(1-\gamma)$$

and

$$\sigma_1 = \frac{2\gamma}{3-\gamma}\left\{\frac{1}{(3-\gamma)^2} + \frac{1}{(1-\gamma)^2}\right\}^{-1}\left\{\frac{1+\frac{1}{24}\gamma^2}{(3-\gamma)^2} + \frac{\frac{1}{4}\gamma^2}{(1-\gamma)^2}\right\},$$

$$\sigma_2 = \frac{2\gamma}{1-\gamma}\left\{\frac{1}{(3-\gamma)^2} + \frac{1}{(1-\gamma)^2}\right\}^{-1}\left\{\frac{\frac{1}{36}\gamma^2}{(3-\gamma)^2} + \frac{1+\frac{3}{8}\gamma^2}{(1-\gamma)^2}\right\}. \tag{4.6}$$

In order to keep the outer mass on the turntable it is necessary for $r_2 < L$ and as $r_2 = \zeta_1 + \zeta_2$, the maximum possible value of r_2 is, from equation (4.5)

$$\frac{1}{6}\frac{(3+\gamma)}{(3-\gamma)}L + \frac{1}{2}\frac{(1+\gamma)}{(1-\gamma)}L,$$

and so for $r_2 < L$

$$\gamma < \frac{7-\sqrt{34}}{5} \quad (\approx 0{\cdot}234). \tag{4.7}$$

For Ω (and hence $\omega^{(0)}$) being such that as γ rises from 0 to 0·234, σ_1 rises (monotonically) from 0 to 0·01422 whilst σ_2 rises from 0 to 0·5785. This means that the coupling of the motions has very little effect upon the larger of the natural frequencies but has a greater effect upon the smaller. In both cases, the frequencies are increased by the coupling.

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APPENDIX A

It is required to show that there is only one solution to equations (2.2) and (2.3) for R and Ω with $R > 0$ and hence $0 \leqslant \Omega < A^{1/2}$ (consider Ω to be positive, as the problem is analogous for negative Ω). Set

$$\Omega = A^{1/2}x,$$

then a solution x is required within $0 < x < 1$. Equation (2.2) is now

$$R = \frac{\hat{r}}{(1-x^2)},$$

and equation (2.3) is

$$h = \left(1 + \frac{mR^2}{I}\right)x,$$

where $h = H/IA^{1/2}$ (> 0).

Therefore, x satisfies the equation

$$h = x\left[1 + \frac{d}{(1-x^2)^2}\right] \tag{A.1}$$

for $d = m\hat{r}^2/I > 0$, and so the roots of the quintic equation

$$f(x) = dx - (h-x)(1-x^2)^2 = 0 \tag{A.2}$$

are required.

Firstly, it is observed that from equation (A.1) for any real root, then $x < h$. From equation (A.2) $f(0) = -h < 0$ and $f(1) = d > 0$ and it follows that there is at least one real root in $0 < x < 1$. Now $f'(x) = d + (1-x^2)^2 + 4x(h-x)(1-x^2)$ and so at any root of f then $f' > 0$ as $h > x$ there. If there were more than one root in $0 < x < 1$ then, as f rises from negative values at $x = 0$ to positive values at $x = 1$, at the smallest root $f' > 0$ as required. However, $f' < 0$ at a second root which cannot be so. It is concluded that there is one and only one value of x in $0 < x < 1$ and hence a unique equilibrium position and rotation rate.

APPENDIX B

The characteristic equation associated with the matrix $\mathbf{\Phi}$, as given by equation (3.12) is

$$f(\phi) = \det(\mu_i^2\delta_{ij} + \rho_j - \phi^2\delta_{ij}) = 0,$$

where the $\mu_i^2 = \lambda_i^2 - \Omega^2$ are taken to be ordered, i.e., $\mu_1^2 < \mu_2^2 < \cdots < \mu_N^2$, and

$$\rho_j = \frac{4\Omega^3}{H} M_j\xi_j^2 = 4\Omega^2\,\frac{M_j\xi_j^2}{(I + \sum M_i^2\xi_i^2)}\,.$$

It is noted that $\rho_j > 0$ for all j. By use of the properties of determinants it is possible to write

$$f(\phi) = \left(1 + \sum \frac{\rho_j}{(\mu_j^2 - \phi^2)}\right) \Pi(\mu_j^2 - \phi^2).$$

It immediately follows that there is a sign change for f between each pair μ_j and μ_{j+1} and one further change in $\phi > \mu_N$. This result relies upon the $\rho_j > 0$. It is concluded that the "coupled" eigenvalues ϕ_j^2 interlace the "uncoupled" values μ_j^2, that is

$$\mu_j^2 < \phi_j^2 < \mu_{j+1}^2 \quad \text{for } j = 1, 2, \ldots, N-1$$

and

$$\mu_N^2 < \phi_N^2.$$